Solution Set: CCCN 2019

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1.1. show that $var(X) = E[X^2] - E[X]^2$

$$var(X) = E[(X - E[X])^2]$$

$$= E[(X^2 - 2X E[X] + E[X]^2)]$$

$$= E[X^2] - 2E[X]E[X] + E[X]^2$$

$$= E[X^2] - E[X]^2$$

1.2. $X \sim N(0, \sigma_X^2)$, $Z \sim N(0, \sigma_Z^2)$

note that $E[X] = E[Z] = 0$; $E[X^2] = \sigma_X^2$, $E[Z^2] = \sigma_Z^2$

find an expression for $var(Y)$, where $Y = X + Z$

$$var(Y) = var(X+Z)$$

$$= E[(X+Z)^2] + E[X+Z]^2 \quad \text{(using previous result)}$$

$$= E[(X^2 + 2XZ + Z^2)] + (E[X] + E[Z])^2$$

($E[X] \to 0$, $E[Z] \to 0$)

$$= E[X^2] + 2E[XZ] + E[Z^2]$$

$$2E[XZ] = 2E[X]E[Z] \text{ for independent variables}$$

$$= 0$$

$$= E[X^2] + E[Z^2]$$

$$= \sigma_X^2 + \sigma_Z^2$$

$$\boxed{var(Y) = \sigma_X^2 + \sigma_Z^2}$$

1.3 show that $I(R;S) = H(R) - H(R|S)$

$$I(R;S) = \sum_r \sum_s p(r,s) \log\left(\frac{p(r,s)}{p(r)p(s)}\right)$$

$$= \sum_r \sum_s p(r,s) \left[\log\left(\frac{p(r,s)}{p(s)}\right) - \log p(r)\right]$$

$$= \underbrace{\sum_r \sum_s p(r,s) \log\left(\frac{p(r,s)}{p(s)}\right)}_{-H(R|S)} - \sum_r \sum_s p(r,s) \log p(r)$$

$$= -H(R|S) - \sum_r \log p(r) \underbrace{\sum_s p(r,s)}_{p(r)}$$

$$= -H(R|S) \underbrace{- \sum_r \log p(r) \cdot p(r)}_{H(R)}$$

$$= H(R) - H(R|S)$$

1.4 for a deterministic encoder, $I(R;S) = H(R) = -\sum_{i=1}^{N} p(r_i) \log p(r_i)$

for $N=2$, $p(r_1) + p(r_2) = 1 \Rightarrow p(r_1) = p$; $p(r_2) = 1-p$

we can write the mutual information as:

$$I = -p\log p - (1-p)\log(1-p)$$

we want to solve $\frac{\partial I}{\partial p} = 0$ to find the value of p that maximizes I

1.4 continued

$$\frac{dI}{dp} = -\left[p \cdot \frac{1}{p} + \log p + (1-p)\cdot \frac{1}{(1-p)}(-1) + (-1)\log(1-p)\right]$$

$$= -\left[1 + \log p - 1 - \log(1-p)\right]$$

$$= -\log p + \log(1-p)$$

$$\frac{dI}{dp} = 0 \Rightarrow \log p = \log(1-p)$$

$$\Rightarrow p = 1-p$$

$$\Rightarrow p = 1/2 \quad (\text{so } p(r_1) = p(r_2) = 1/2)$$

$$I(p=1/2) = -\sum_{i=1}^{2} \frac{1}{2}\log_2\left(\frac{1}{2}\right)$$

$$= -\log_2\left(\frac{1}{2}\right)$$

$$= \log_2(2)$$

$$= 1 \text{ bit}$$

1.5 if $p(r_i) = \frac{1}{N} \ \forall i$ $\Rightarrow$ all assignments of spike counts to responses produce the same average spike count, because all responses are used w/ equal frequency on average

if $p(r_1) = \frac{1}{3}$, $p(r_2) = \frac{1}{3}$, $p(r_{i>2}) = \frac{1}{3(N-2)}$

$\Rightarrow$ the best strategy (that minimizes avg. spike count) is to assign 0, 1 spikes to r_1, r_2, and assign $k>1$ spikes to $r_{i>2}$

Problem 2.4

Determine the parameters of the nonlinearity that maximize the entropy of the distribution of spike counts.

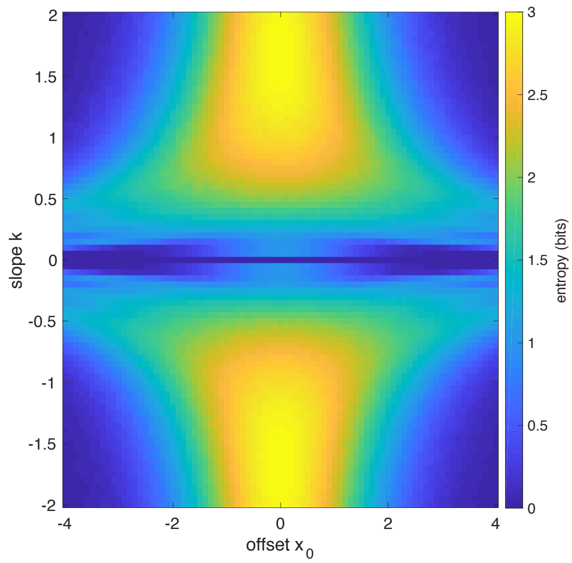

Figure 1: **Entropy surface for varying encoding parameters.**

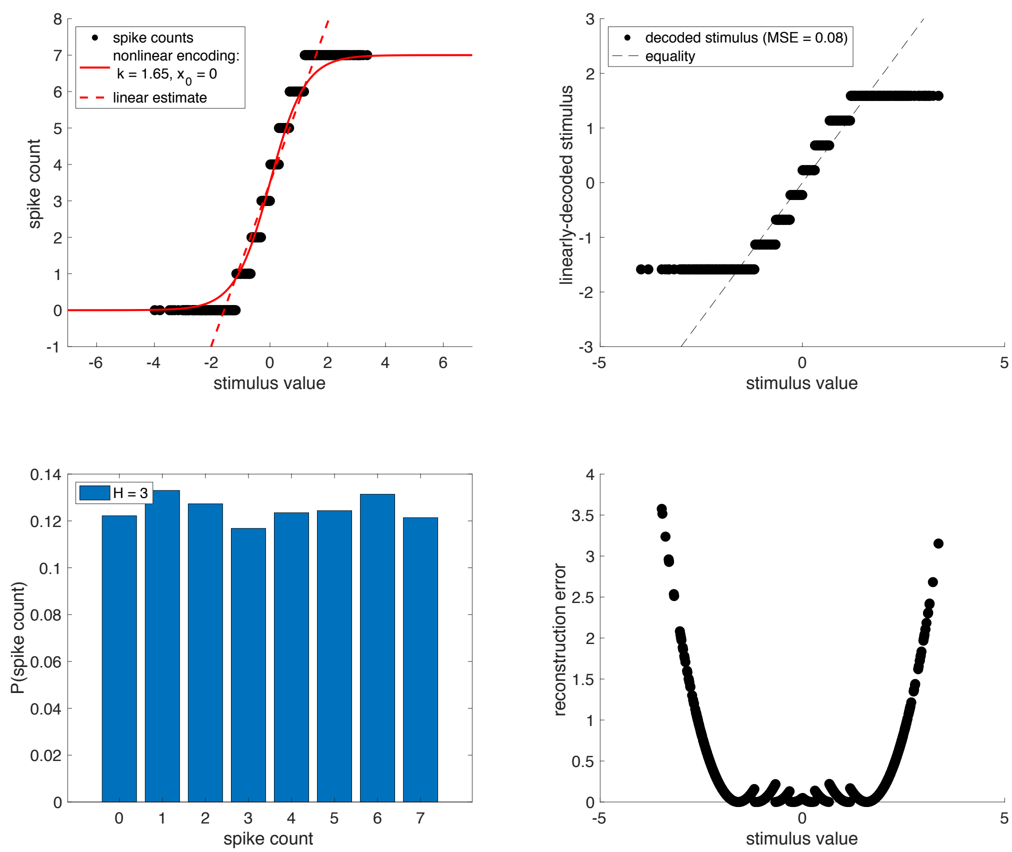

Figure 2: **Nonlinearity that maximizes entropy.**

Problem 2.5

Find the parameters of the nonlinearity that minimize the average reconstruction error $\langle (s - \hat{s})^2 \rangle$.

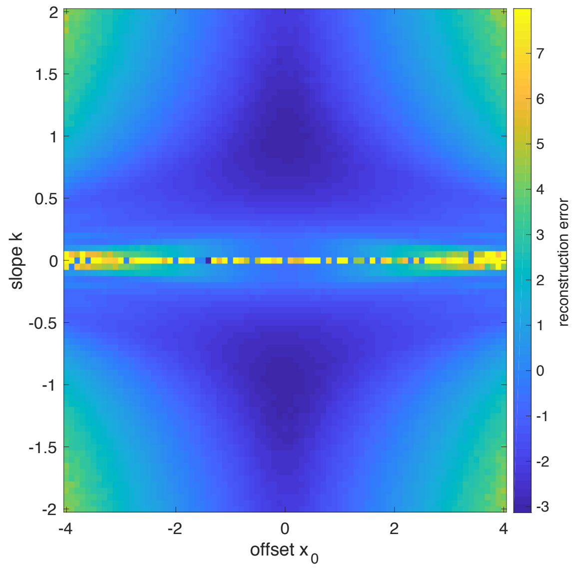

Figure 3: **Error surface for varying encoding parameters.**

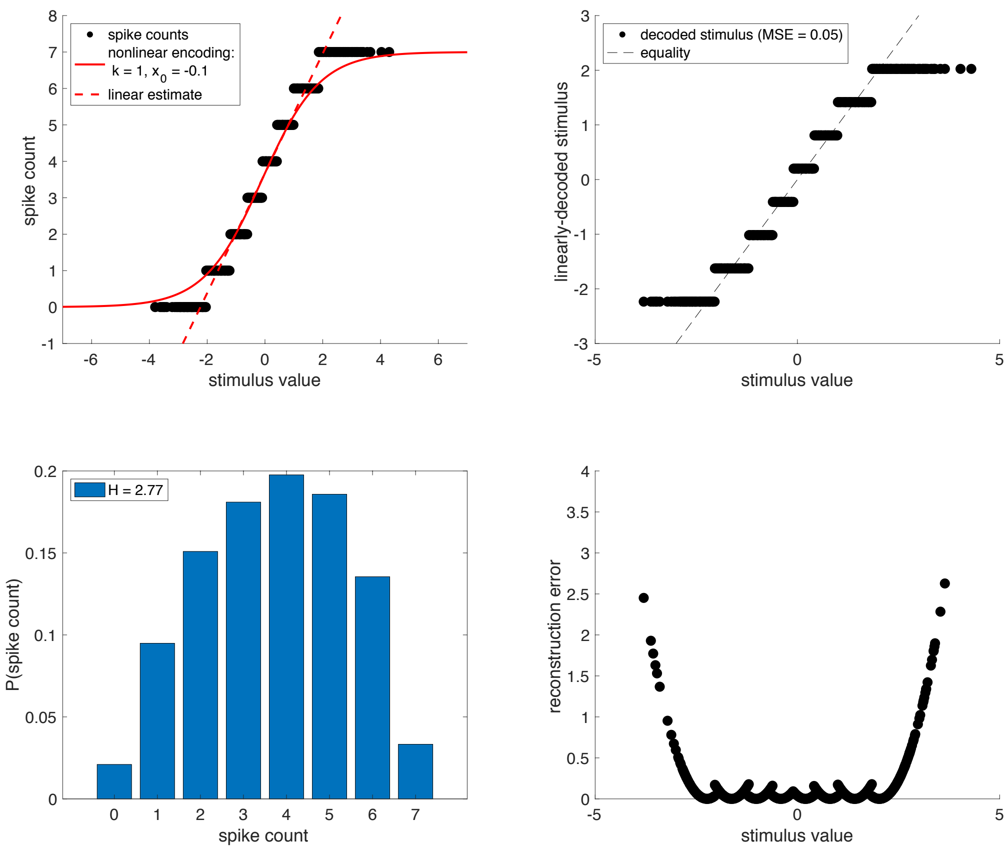

Figure 4: **Nonlinearity that minimizes reconstruction error.**

Problem 2.6

Vary the mean μ of your stimulus distribution (keeping the variance fixed), and compute the optimal values of k and x_0 as a function of μ. Repeat, varying the variance σ^2 while keeping the mean fixed.

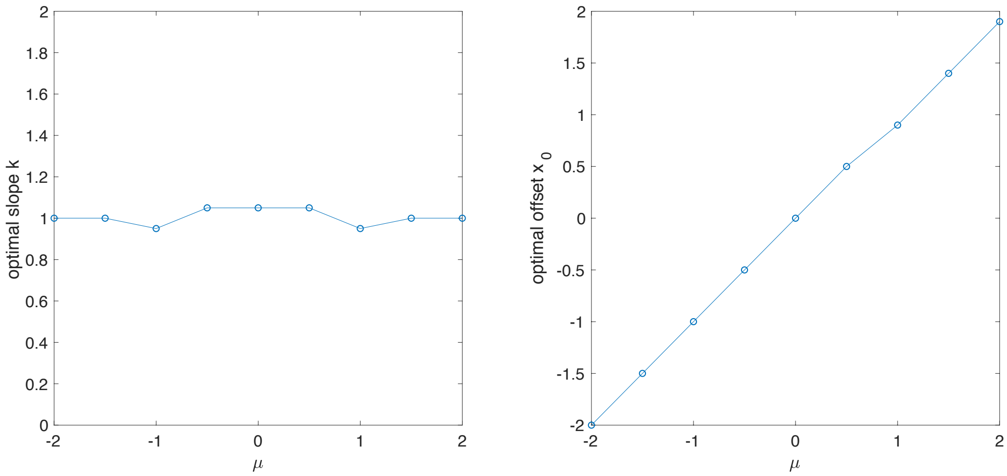

Figure 5: **Optimal nonlinearity parameters for different stimulus means.**

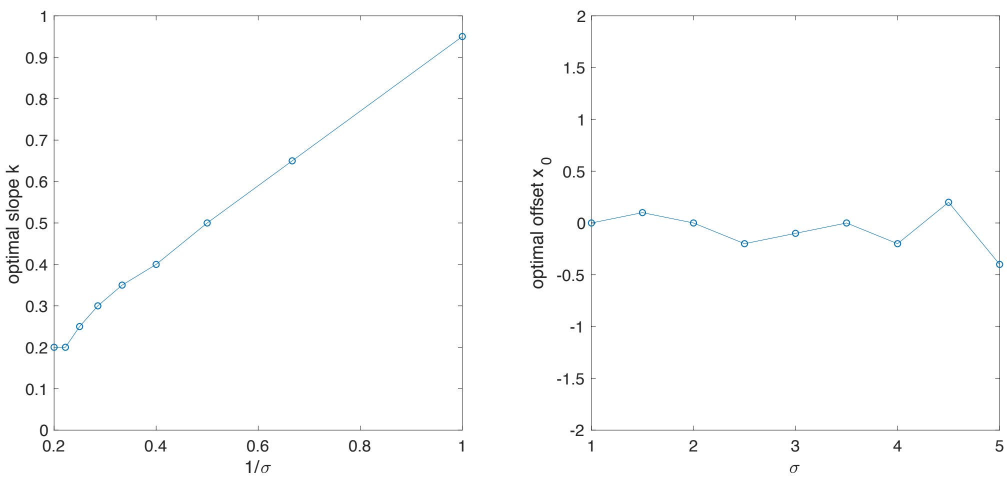

Figure 6: **Optimal nonlinearity parameters for different stimulus variances.**

Problems 2.7-2.8

Find the optimal nonlinearity parameters as a function of input noise, σ_{in}^2, and output noise, σ_{out}^2.

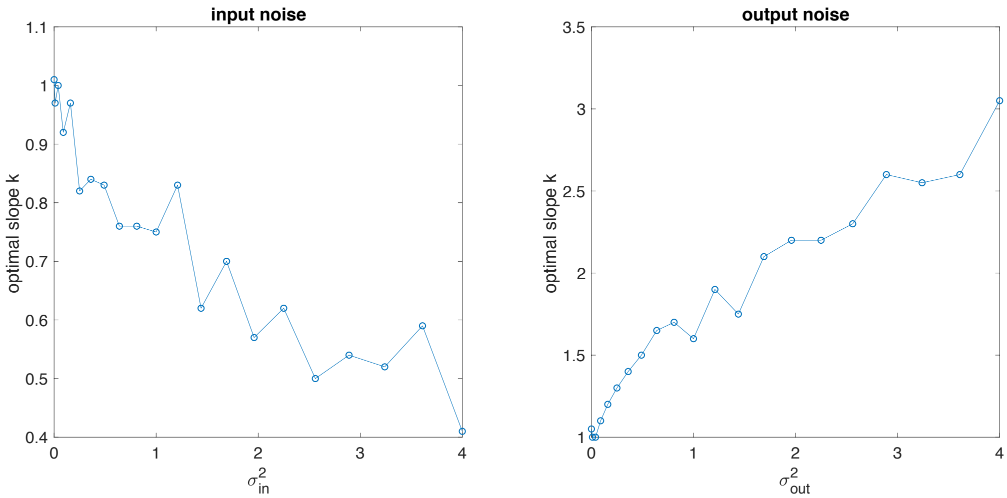

Figure 7: **Optimal nonlinearity parameters for a different levels of input noise.**

Problems 2.9

Compute the average reconstruction error as a function of the mismatch between the stimulus mean μ and the system's internal estimate of the mean $\hat{\mu}$. Repeat for the mismatch between the stimulus variance σ^2 and the system's internal estimate of the variance $\hat{\sigma}^2$.

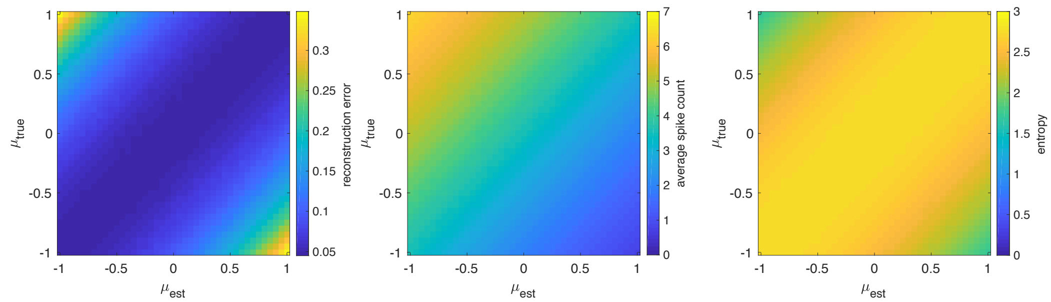

Figure 8: **Reconstruction error, average spike count, and entropy as a function of the mismatch between the true and estimated stimulus mean.**

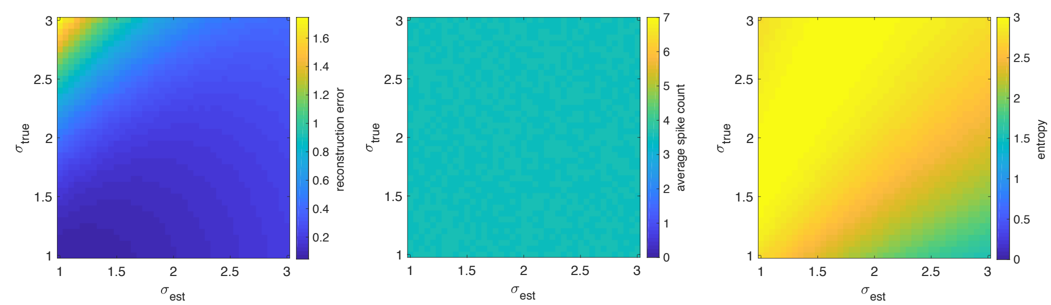

Figure 9: **Reconstruction error, average spike count, and entropy as a function of the mismatch between the true and estimated stimulus variance.**